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Inertial modes with large azimuthal wavenumbers in an axisymmetric container

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Free oscillations are considered of a fluid rotating with constant angular velocity $\Omega\hat{\mathbf{z}}$ in a rigid axisymmetric container. Modes are sought that vary rapidly in an axial (r, z) plane with a length scale $O(n^{-1})$ times that of the container, where $n \gg 1$. The azimuthal wavenumber $k > 0$ is also taken to be large. The modulated wave modes postulated (represented as in (4.1)) prove to have a quiescent zone near the axis. Elsewhere their pressure is of a uniform order of magnitude. Their velocity however is locally magnified by a factor $O(n)$ near the critical circles. It is argued that for $k/n \ll 1$ the modulated waves eligible as modes in smooth, convex containers are of two kinds; one, which generally occurs for continuous frequency bands, being singular and indeterminate; the other being like the modes in a sphere. Modes of the second kind are determined for eigenfrequencies $\omega \simeq \sqrt{2}\,\Omega$ for containers whose axia lcross-sections are symmetrical about $z = 0$ and about $r = \pm z$.

1. Introduction

Although the free linear oscillations of finite bodies of uniformly rotating inviscid fluid have been studied a long time, little systematic is known about them because of their abnormal spectral properties.

The abnormality results from the equation (Poincaré 1885) for the spatial variation of the modes being hyperbolic, and it is clearly displayed when the modes vary in only two dimensions. For two-dimensional modes, the stream function $\psi e^{i\omega t}$ has the general solution $\psi = \psi_0(x-\alpha z) - \psi_0(x+\alpha z)$, where $\alpha(\omega)$ is a constant and $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$ is the primary rotation. The existence of modes then depends on the geometry of the paths formed by the rays $x \pm \alpha z =$ constant after successive reflections at the fluid's boundary. For example, continuous modes exist in a smooth convex rigid container if, and only if, all the ray paths close (Schaeffer 1975). Singular modes are also possible, as we shall see later. Modes for specific geometries have been given by Barcilon (1968) and Franklin (1972). These modes are presented as approximations to the axisymmetric modes in a thin torus, but they are in effect two-dimensional. Results for forced motion follow from general theorems for hyperbolic equations with Dirichlet boundary conditions (Bourgin & Duffin 1939; John 1941). In all these cases the topology of the ray paths after multiple reflections is crucial, and this topology can be highly sensitive to changes of the boundary or of the ray directions.

The complexity evident in two dimensions prompts closer scrutiny of modes in axisymmetric bodies of fluid, which are more commonly of physical interest. The only relevant exact eigensolutions known are for cylinders and cylindrical annuli (Kelvin

1880) and for spheres and spheroids (Bryan 1889; Hough 1895; Kudlick 1966). These modes have continuous velocities and discrete densely distributed eigenfrequencies, but they may not be typical of other axisymmetric geometries.

Further progress has been made by restricting attention to modes with a small length scale. Rapidly varying modes in a thin spherical annulus have been studied by Stewartson & Rickard (1969) and later, by ray methods, by Bretherton (1964) and Stewartson (1971, 1972). Rapidly varying modes have also been studied in a geomagnetic context by Malkus (1967), Roberts (1968) and Busse (1970). Malkus† investigated the high-order asymmetric modes in a sphere in relation to the westward drift of the earth's magnetic field. Roberts and Busse investigated rapidly varying asymmetric modes, though for a diffusive, Boussinesq fluid, both in a sphere and in a cylinder, in relation to the geodynamo.

The present paper deals with arbitrary axisymmetric geometries, by ray methods. The azimuthal wavenumber k is taken to be large. An earlier paper (Wood 1977*b*) dealt with the case $k = O(1)$. The main differences here stem from the curvature of the ray paths in an axial plane. (When $k = O(1)$, these paths are effectively straight.) One consequence is a still zone about the axis. A second is that the patterns of the ray paths after multiple reflections at the boundary are less predictable, especially when k/n is not small.

The conclusions are collected at the end of the paper.

2. Formulation

We consider oscillations in finite, rigid axisymmetric containers S that rotate with constant angular velocity $\mathbf{\Omega}$ about their axis of symmetry. The fluid is assumed to be inviscid and of constant density. The velocity $\mathbf{u}$ of the fluid relative to S is taken to be small relative to $\mathbf{\Omega}\times\mathbf{r}$ and to vary as

$$\mathbf{u} = \mathrm{Re}\,\{\mathbf{U}(r,z)\,e^{i(\omega t+k\phi)}\}, \quad k > 0, \tag{2.1}$$

the azimuthal angle ϕ being measured relative to (a frame fixed in) S, whilst the corresponding perturbation pressure p due to the motion relative to S is taken to vary as

$$p = \mathrm{Re}\{r^{-\frac{1}{2}}q(r,z)\,e^{i(\omega t+k\phi)}\}. \tag{2.2}$$

This pressure p satisfies Poincaré's equation

$$\frac{\partial^2}{\partial t^2}\nabla^2 p + 4\Omega^2\frac{\partial^2 p}{\partial z^2} = 0 \tag{2.3}$$

(Poincaré 1885). So, the pressure amplitude q satisfies the equation

$$q_{rr} - \alpha^2 q_{zz} = \beta r^{-2} q, \tag{2.4}$$

where

$$\alpha = (4\Omega^2\omega^{-2}-1)^{\frac{1}{2}}, \quad \beta = k^2 - \tfrac{1}{4}. \tag{2.5}$$

The velocity amplitude $\mathbf{U} = (U,\ V,\ W)$ is related to q by

$$U = i\omega r^{-\frac{1}{2}}[q_r + (2\Omega\omega^{-1}k - \tfrac{1}{2})\,qr^{-1}]/(\omega^2 - 4\Omega^2), \tag{2.6}$$

$$V = 2\Omega r^{-\frac{1}{2}}[q_r + \tfrac{1}{2}(\omega\Omega^{-1}k - 1)\,qr^{-1}]/(4\Omega^2 - \omega^2), \tag{2.7}$$

$$W = i\omega^{-1}r^{-\frac{1}{2}}q_z. \tag{2.8}$$

† Certain omitted modes were given subsequently by the present author (Wood 1977*a*, *b*).

Hence, the vanishing of the normal velocity at the boundary, defined say by $r = S(z)$, implies that

$$q_r + \alpha^2 S' q_z + \gamma q r^{-1} = 0 \quad \text{on } S, \tag{2.9}$$

where

$$\gamma = 2\Omega\omega^{-1}k - \tfrac{1}{2}. \tag{2.10}$$

For brevity, **U** and q will be called the velocity and pressure. Eigenmodes exist only if $0 \leqslant |\omega| \leqslant 2\Omega$ (cf. Greenspan 1968). So α is real, and q can be taken to be real.

3. Free oscillations in cylinders and spheres

As a guide to the possible behaviour of rapidly varying modes in other geometries, a preliminary discussion is given here of the high-order modes in a cylindrical annulus and in a sphere.

The free oscillations in a rigid cylindrical annulus, $0 < R_1 < r < R_2$, $0 < z < \pi$, are determined by

$$q = n^{\frac{1}{2}} r^{\frac{1}{2}} [J_k(\alpha n r) - B Y_k(\alpha n r)] \cos nz, \tag{3.1}$$

where

$$B = \frac{\alpha\omega n R_2 J_k'(\alpha n R_2) + 2\Omega k J_k(\alpha n R_2)}{\alpha\omega n R_2 Y_k'(\alpha n R_2) + 2\Omega k Y_k(\alpha n R_2)} = \frac{\alpha\omega n R_1 J_k'(\alpha n R_1) + 2\Omega k\, J_k(\alpha n R_1)}{\alpha\omega n R_1 Y_k'(\alpha n R_1) + 2\Omega k Y_k(\alpha n R_1)}. \tag{3.2}$$

For $\alpha r n \gg 1$ and fixed values of $k/\alpha r n \neq 1$, the pressure is represented to a first approximation by

$$q \sim c_0 |1 - \lambda^2 \alpha^{-2} r^{-2}|^{-\frac{1}{4}} \operatorname{Re}[A_- e^{in\nu_-} + A_+ e^{in\nu_+}], \tag{3.3}$$

where c_0 is a constant $O(1)$,

$$\lambda = k/n, \tag{3.4}$$

$$in\nu_\pm = iz \mp (\lambda^2 - \alpha^2 r^2)^{\frac{1}{2}} \pm \lambda \cosh^{-1}(\lambda/\alpha r), \quad A_- = 1, \quad A_+ = 2B \quad \text{for} \quad \alpha r < \lambda, \tag{3.5}$$

$$n\nu_\pm = \mp z + (\alpha^2 r^2 - \lambda^2)^{\frac{1}{2}} - \lambda \cos^{-1}(\lambda/\alpha r) - \tfrac{1}{4}\pi n, \quad A_- = A_+ = 1 + iB, \quad \text{for} \quad \alpha r > \lambda. \tag{3.6}$$

For wavenumber ratios $(\lambda =)\, k/n < \alpha R_1$, the pressure is uniformly $O(1)$. But, for ratios $k/n > \alpha R_1$, the pressure is $O(1)$ only where $k/\alpha n < r < R_2$ and (apart from a thin transition zone) is exponentially small $O(e^{-|O(n)|})$ elsewhere. Thus (for $k/n > \alpha R_1$) the wave is effectively confined to the outer annulus $k/\alpha n < r < R_2$. This wave-trapping occurs in a circular cylinder whenever $n/k = O(1)$.

The eigenvalues α must be greater than k/nR_2, because (3.2) has no solutions if the exponential approximation (3.5) applies at $r = R_1$ and R_2 (or at $r = R_2$ alone if $R_1 = 0$). So the eigenfrequencies are bounded by

$$|\omega| < 2\Omega(1 + k^2/n^2 R_2^2)^{-\frac{1}{2}}, \tag{3.7}$$

and the quiescent zone (where the pressure is $O(e^{-|O(n)|})$) never extends to the outer cylinder.

As one would expect, the presence of the rigid boundary in the quiscent zone has little effect on the trapped wave. When a quiescent zone occurs in an annulus, the eigenfrequency and the velocity are determined with an error $O(e^{-|O(n)|})$ by putting $B = 0$ in (3.1) and (3.2) and ignoring the boundary condition at $r = R_1$. To this accuracy, the same trapped wave could also occur in the circular cylinder $0 < r < R_2$, the only significant change to the overall flow being the increased extent (namely $0 < r < k/\alpha n$ instead of $R_1 < r < k/\alpha n$) of the associated quiescent zone.

Lastly, it might be noted that the characteristics ($r \pm \alpha z = \text{const.}$) of the eigenmodes do not always form closed surfaces after repeated reflections at the boundaries and the axis. For the cylindrical annulus, closure occurs only when $\Gamma = \pi^{-1}(R_2 - R_1)\cot\alpha$ is a rational number. However, the eigenequation defines α, and hence Γ, as a continuous function of R_2, for given values of $R_1 \geqslant 0$, k, n and a given number of cells in the radial direction. Hence, as R_2 varies, *both closure and non-closure* occur. When $k = 0$, the same conclusion applies to the paths formed by the trajectories of the group velocity in a meridional plane ($\phi = 0$) after repeated reflections at the boundaries and axis, because the trajectories of these axisymmetric modes coincide with the characteristics.

We turn now to the eigenmodes in a rigid sphere $|\mathbf{r}| < 1$, which are given by

$$q = (\text{constant}) \times r^{\frac{1}{2}} P_n^k(\cos\Theta)\, P_n^k(\cos\Phi) \tag{3.8}$$

(cf. Greenspan 1968), where

$$r\cos\theta = \sin\Theta\sin\Phi, \quad z\sin\theta = \cos\Theta\cos\Phi \tag{3.9}$$

with $0 \leqslant \Theta \leqslant \frac{1}{2}\pi - \theta \leqslant \Phi \leqslant \frac{1}{2}\pi + \theta$. The angle θ here is the semi-angle, $\cot^{-1}\alpha$, of the characteristic cones of Poincaré's equation (2.4). The eigenfrequencies are determined by the equations

$$\omega = 2\Omega\sin\theta, \quad kP_n^k(\sin\theta) = \cos\theta\, d\, P_n^k(\sin\theta)/d\theta. \tag{3.10}$$

For large n and fixed values of $\lambda_0 = k/(n+\frac{1}{2})$,

$$P_n^k(\sin\theta) \sim \begin{cases} c_1(\lambda_0^2 - \cos^2\theta)^{-\frac{1}{4}} \exp[(n+\frac{1}{2})\chi(\sin\theta)], & 0 \leqslant \cos\theta < \lambda_0, \quad (3.11) \\ 2c_1(\cos^2\theta - \lambda_0^2)^{-\frac{1}{4}} \cos[(n+\frac{1}{2})\chi(\sin\theta) - \frac{1}{4}\pi], & \lambda_0 < \cos\theta \leqslant 1, \quad (3.12) \end{cases}$$

where c_1 is a constant which will be taken to be $O(1)$, and

$$\chi(\sin\theta) = \begin{cases} \cosh^{-1}[(1-\lambda_0^2)^{-\frac{1}{2}}\sin\theta] - \lambda_0\cosh^{-1}[\lambda_0(1-\lambda_0^2)^{-\frac{1}{2}}\tan\theta], \\ \qquad\qquad 0 \leqslant \cos\theta < \lambda_0, \quad (3.13) \\ \cos^{-1}[(1-\lambda_0^2)^{-\frac{1}{2}}\sin\theta] - \lambda_0\cos^{-1}[\lambda_0(1-\lambda_0^2)^{-\frac{1}{2}}\tan\theta], \\ \qquad\qquad \lambda_0 < \cos\theta \leqslant 1 \quad (3.14) \end{cases}$$

(Thorne 1957). A short calculation shows that the exponential approximation (3.11) provides no solutions to the eigenequation (3.10). So, the eigenvalues of θ are such that $\cos\theta > \lambda_0$ and the eigenfrequencies are such that

$$|\omega| < 2\Omega[1 - k^2(n+\tfrac{1}{2})^{-2}]^{\frac{1}{2}}. \tag{3.15}$$

Under these conditions, the eigenvalues of θ are given to first order by

$$\tan[(n+\tfrac{1}{2})\chi(\sin\theta) - \tfrac{1}{4}\pi] = -\lambda_0(\cos^2\theta - \lambda_0^2)^{-\frac{1}{2}}. \tag{3.16}$$

The corresponding approximations for the pressure q for large n and fixed values of $k/(n+\frac{1}{2})$ are given by

$$q \sim 2c_1^2|\sin^2\Theta - \lambda_0^2|^{-\frac{1}{4}}(\sin^2\Phi - \lambda_0^2)^{-\frac{1}{4}}\,\mathrm{Re}\,[e^{in\nu_+} + A_- e^{in\nu_-}], \tag{3.17}$$

where

$$n\nu_+ = (n+\tfrac{1}{2})[\chi(\cos\Phi) - i\chi(\cos\Theta)] - \tfrac{1}{4}\pi, \quad A_- = 0, \quad \sin\Theta < \lambda_0, \tag{3.18}$$

$$n\nu_\pm = (n+\tfrac{1}{2})[\chi(\cos\Theta) \pm \chi(\cos\Phi)] - \tfrac{1}{2}\pi, \quad A_- = i, \quad \sin\Theta > \lambda_0. \tag{3.19}$$

The real part $(n+\frac{1}{2})\chi(\cos\Theta)$ of the exponent $in\nu_+$, given by (3.18) for $\sin\Theta < \lambda_0$, vanishes when $\sin\Theta = \lambda_0$ and decreases as Θ decreases. So, when $n/k = O(1)$, wave trapping recurs. Transition now occurs near where $\sin\Theta = \lambda_0$ at the spheroid

$$r^2\lambda_0^{-2}\cos^2\theta + z^2(1-\lambda_0^2)^{-1}\sin^2\theta = 1. \tag{3.20}$$

The pressure q is $O(1)$ farther from the axis than the thin transition layer and is $O(e^{-|O(n)|})$ nearer to the axis. The spheroid meets the sphere where $r = \lambda_0$. So, since $\lambda_0 < \cos\theta$, the entire fluid axis $r = 0$, $|z| < 1$ lies in the quiescent zone; whilst the critical circles $r = \cos\theta$, $z = \pm\sin\theta$, at which the characteristic cones graze the sphere, lie in the trapped wave. The insensitivity of the trapped wave between two cylinders to the presence of the inner cylinder suggests that the trapped wave in a sphere (where $\sin\Theta > \lambda_0$) may only be slightly affected by imposing a rigid axisymmetric boundary in its quiescent interior (where $\sin\Theta < \lambda_0$). In particular, each of the eigenmodes in the sphere with $\lambda_0 > R_3\cos\theta$ might be surmized to have a counterpart with almost the same eigenfrequency and trapped motion in a spherical annulus $R_3 < |\mathbf{r}| < 1$.

The velocity amplification that arises near the critical circles when $k = O(1)$ (Wood 1977*b*) is reproduced in a similar way when $k \gg 1$. Consider the critical circle (C_N), $r = \cos\theta$, $z = \sin\theta$, at which $\Theta = \Phi = \frac{1}{2}\pi - \theta$. Because of the relative crowding of characteristic cones after their reflection near C_N parallel to the tangential cone at C_N, the quantity $\partial(\Phi-\Theta)/\partial r$ becomes infinite at C_N. Consequently at C_N the gradient $\nabla\nu_- = \infty$ and the velocity is large. On closer examination, we find that the velocity amplitude near C_N is given, to a first approximation, by

$$\mathbf{U} = c_1^2\,\Omega^{-1}\sec^2\theta[-i\sin\theta, 1, i\cos\theta]\frac{(n+\frac{1}{2})\sin\{(n+\frac{1}{2})\sec\theta\,(\cos^2\theta-\lambda_0^2)^{\frac{1}{2}}(\Theta-\Phi)\}}{\Theta-\Phi}. \tag{3.21}$$

Hence the velocity amplitudes at distances $O(n^{-2})$ from the critical circle in a direction normal to the boundary are a factor $O(n)$ greater than the velocity amplitudes elsewhere in the critical zone.

A generalization of the eigenfrequency equation (3.16) to a non-spherical configuration is given in §5.

4. Modulated waves in axisymmetric containers

Free oscillations are now considered for smooth convex axisymmetric containers ($r = S(z)$, $|z| < 1$) of arbitrary shape. As above, approximations are sought for modes that vary rapidly in each axial plane (with a length scale n^{-1}) and vary rapidly (as $e^{ik\phi}$) with the azimuthal angle ϕ. The ratio $\lambda = k/n$ is taken to be fixed. For simplicity, containers with vertices at $z = 1$ or -1 are excluded. Complications that might arise if a characteristic and its successive reflections converge progressively into a vertex or corner are thus avoided. (The cylindrical containers of §3 do not give rise to such difficulties because of their sides being either parallel or to normal to $\mathbf{\Omega}$.)

The assumption that the modes can be represented as a pair of modulated waves (as in (4.1)) is seen to lead to generalizations of the salient features found above for the sphere and the cylinders. In particular the fluid near the axis is again almost still. Also the velocity amplitudes near the critical circles are again found to be a factor $O(n)$ greater than elsewhere in the region of the modulated waves. To deal with the decay of the modulated waves at the edge of the axial zone, the variation of $r^{\frac{1}{2}}q$ along the axis is assumed to have a quasi-periodic representation (as in (4.13) and (4.14)). This axial representation is matched to the modulated waves via an integral solution for the flow in the double-cone ($|z| < 1-\alpha r$), which forms the domain of dependence of the hyperbolic equation (2.4) on the axial data. The method requires the double-cone to overlap the region of modulated waves. However, the evidence points to the modulated waves penetrating to within $O(k/n)$ of the axis. So an overlap can be anticipated for sufficiently small values of k/n. The integral then yields simply an approximation for the velocity in the transition zone and an order-of-magnitude estimate for the velocity in the quiescent zone. A simple argument extends these results to the parts of the transition zone and quiescent zone outside the double-cone.

Determination of the phases ($\sigma_j(r,z)$) of the modulated waves is, in general, hampered by the intricacy of the patterns of the ray paths (for the meridional component) of the group velocity formed after multiple reflections at the boundaries. Consequently the phases have been determined explicitly (at the boundary) only for configurations with relatively simple ray patterns. A general method of calculating the phases $\sigma_j(r,z)$ for $k \ll n$ is proposed (in § 5), which appears feasible (in principle) provided that no singularities prove to emanate from the poles or the critical circles. This contingency does not arise for the particular case treated explicitly or for the sphere, but its absence has not been generally established. To the extent that the feasibility of determining $\sigma_j(r,z)$ and the associated phase $f(z)$ (in (4.14)) has not been established, the validity of the modulated wave representation, and the associated assumptions, remain unproved, and the conclusions below should be regarded as tentative.

(a) *Oscillatary zone: ray method*

With the sphere and cylinder as a guide (and the cautionary remarks above in mind), we postulate a region of modulated waves in which the pressure q is represented by

$$q \sim n^{-\frac{1}{2}} \operatorname{Re}\{\sum_{j=1,2} A_j(r,z,n)\, e^{in\sigma_j(r,z)}\}, \tag{4.1}$$

where the phases $\sigma_j(r,z)$ are real and the amplitudes are $O(1)$ and have asymptotic expansions in descending integral powers of n. For definiteness the signs of the σ_j are chosen so that $\partial\sigma_j/\partial z \geqslant 0$. Further, we assume that each of the modulated waves (for $j = 1$ or 2) in (4.1) satisfies the equations of motion independently. Such assumptions underlie the so-called ray method developed for geometrical diffraction (Keller 1958).

The properties of three-dimensional modulated (inertial) waves are worth recalling here. If such a wave is represented to leading order by

$$p = A(\mathbf{x}) \exp[i(n\Sigma(\mathbf{x}) + \omega t)], \tag{4.2}$$

its phase Σ and amplitude A satisfy the dispersion relation

$$\omega^2\boldsymbol{\kappa}^2 = 4(\boldsymbol{\kappa}.\boldsymbol{\Omega})^2 \tag{4.3}$$

and the amplitude equation

$$\operatorname{div}(A^2\Sigma_z^2\,\mathbf{V}) = 0, \tag{4.4}$$

where $\boldsymbol{\kappa} = n\nabla\Sigma$ is the local wavenumber and

$$\mathbf{V} = \nabla_\kappa\,\omega = (\Sigma_x, \Sigma_y, -\alpha^2\Sigma_z)/n\alpha\,(d\alpha/d\omega)\Sigma_z^2 \tag{4.5}$$

is the local group velocity. The wavenumber $\boldsymbol{\kappa}$ and the group velocity $\mathbf{V}$ are constant

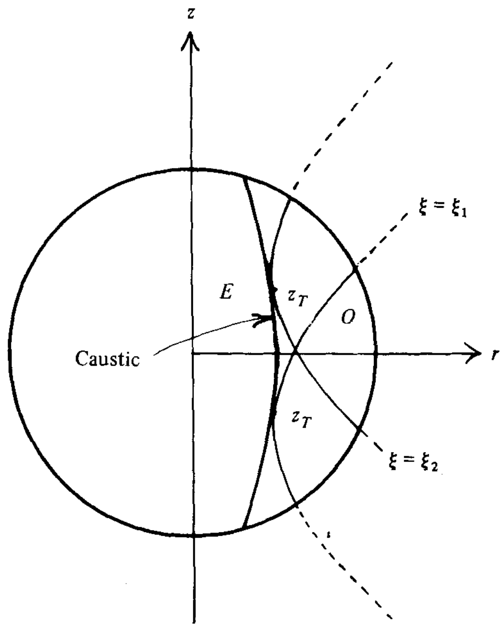

FIGURE 1. A typical oscillatory zone O and quiescent zone E, separated by a caustic, and typical trajectories ($\xi = \xi_1, \xi_2$) of the phase equation (4.6).

along each straight ray in the direction of **V** (since the dispersion relation (4.3) is independent of **x** and t). Further, the phase Σ is constant on each such ray (because (4.3) is a homogeneous quadratic form in the components of $\boldsymbol{\kappa}$). Finally, since $\Sigma_z^2 \mathbf{V}$ is constant along each ray, the conservation equation shows that A^2 is inversely proportional to the cross-sectional area of any thin tube bounded by the rays.

For the modulated waves appropriate to the axial symmetry in (4.1), the phase Σ becomes $\sigma_j(r, z) + \lambda\phi$ and the phase and amplitude equations of the three-dimensional waves reduce to

$$\sigma_r^2 - \alpha^2\sigma_z^2 + \lambda^2 r^{-2} = 0, \tag{4.6}$$

$$(\sigma_r A^2)_r - \alpha^2(\sigma_z A^2)_z = 0, \tag{4.7}$$

where $\sigma = \sigma_j$, $A = A_j$ and $j = 1$ or 2. It follows that the trajectories of the component of the group velocity in the meridional plane (ϕ = const.) are the hyperbolae

$$(z-\xi)^2 = \alpha^2 r^2 - \lambda^2\zeta^{-2}, \quad \xi = \text{const.}, \quad \zeta = \text{const.}, \tag{4.8}$$

where $\zeta = \phi_z$, and that the phase σ and amplitude A are given by

$$\sigma = \eta + \lambda \cos^{-1}\{(\xi - z)/\alpha r\}, \quad A = r^{\frac{1}{2}} A^0 |\zeta(z-\xi)\,(d\xi/d\zeta) + \lambda^2\zeta^{-2}|^{-\frac{1}{2}}, \tag{4.9}$$

where the $\cos^{-1}\{(\xi - z)/\alpha r\}$ is in the range $(0, \pi)$ and η and A^0 are constant on each trajectory ξ = constant. Since ϕ_z depends only on ξ, (4.8) and (4.9) imply that

$$\frac{d\eta(\xi)}{d\xi} = \zeta. \tag{4.10}$$

Thus, the phase and, to a leading order, the amplitude of each wave (for $j = 1$ or 2) are determined apart from the two parameters $\eta(\xi)$ and $A^0(\xi)$. In keeping with equation (4.4), the factor $(A/A^0)^2$ in (4.9) is inversely proportional to the distance between neighbouring trajectories. Where two trajectories coalesce, the amplitude A given by (4.9) is infinite and the modulated wave approximation fails. Otherwise the modulated waves, each of which comprises two travelling waves, presumably propagate freely along the trajectories. At the solid boundary, the waves reflect without change of form (as will be plain below). So the oscillatory region must include the entire length lying in the fluid of any trajectory as defined by (4.8) and all its reflections. The assumption of two modulated waves (the least number that allows for reflection) in (4.1) means that there are two trajectories through any point r, z in a meridional plane. So any boundaries in the fluid of the oscillatory region must be envelopes of the trajectories (to within distances $O(1)$).

One such boundary must separate the modulated waves from the z axis. The apex of each (hyperboloid) trajectory ξ = const. is a distance $k/\alpha n\zeta(\xi)$ from the axis, which is non-zero (for $k \gg 1$) if $\omega \neq 0$ (i.e. $\alpha \neq \infty$) and $n\zeta$ is finite. We can assume $\omega \neq 0$ (for $k \gg 1$), since the only steady modes are the swirling modes ($\mathbf{U} = (0, V(r), 0)$) for $k = 0$. If the axial wavenumber $n\zeta$ were infinite, for $\xi = \xi_0$ say, the modulated wave approximation would imply infinite velocities in the fluid along the trajectory ξ_0 = constant. Infinite velocities of this kind do not appear in the sphere or the cylinder though they may possibly occur for certain frequency bands in other containers, as is noted in §6. However, ζ can be infinite, at most on isolated trajectories, by virtue of its definition as $\zeta = \phi_z$. Hence, the oscillatory region is separated from the axis of symmetry by an envelope ($r = r_T(z)$ say) of the trajectories, which meets the axis at most at isolated points (figure 1). The zone $r < r_T(z)$ excluded from the oscillatory region has a radius $O(k/n)$. The excluded axial zone will be presumed to be quiescent, and to some extent this will be confirmed below.

Other modulated waves that are represented as in (4.1) and have the phase $\Sigma = \sigma(r,z)+\lambda\phi$ appropriate to axial symmetry also avoid the axis. We refer here specifically to waves propagating relative to a steady, homogeneous medium that is isotropic in planes z = const., so that the dispersion relation has the form $D(\Sigma_z, \Sigma_x^2+\Sigma_y^2, \omega) = 0$. For such waves, the wavenumber $\boldsymbol{\kappa}(= n\nabla\omega)$ and the group velocity $\mathbf{V}(= \nabla_{\kappa}\omega)$ are again constant on rays in the direction of $\mathbf{V}$ and the components of $\boldsymbol{\kappa}$ and $\mathbf{V}$ in planes z = const. are parallel. The nearest distance to the axis of any ray is thus

$$rV_\phi/(V_r^2+V_\phi^2)^{\frac{1}{2}} = r\kappa_\phi/(\kappa_r^2+\kappa_\phi^2)^{\frac{1}{2}} = k/ng(\sigma_z), \tag{4.11}$$

where $g(\sigma_z)$ represents $n^{-1}(\kappa_r^2+\kappa_\phi^2)^{\frac{1}{2}}$ and is determined by the dispersion relation. Hence, provided that $g(\sigma_z)$ is finite, the modulated waves with $k \gg 1$ again avoid the z axis. For the inertial waves, $g = \alpha\phi_z$. For two-dimensional modes, as say the acoustic modes in a cylinder considered by Lord Rayleigh (1910) in relation to the whispering gallery phenomenon (Keller and Rubinow 1960), g is a constant.

(b) *Quiescent axial zone and transition zone: integral method*

We now consider the axial region not penetrated by the modulated waves. In the expectation that the modulated waves can be determined (in principle) independently of the flow elsewhere, we regard the modulated waves as known and seek approximations that match them. A minor exception to this concerns the phase jump in a modulated wave that passes near to the boundary ($r = r_T(z)$) of the axial region, which is needed in the determination of the modulated waves, the amount of this jump is derived *from* the matching.

To find approximations matching the modulated waves, we use the general solution for the pressure q given by

$$q = r^{\frac{1}{2}} \int_0^{\pi} F(z - \alpha r \cos x) \cos kx \, dx, \tag{4.12}$$

where $F(z)$ is defined (sufficiently) by

$$\lim_{r \to 0} r^{-k-\frac{1}{2}} q = \pi(-\alpha/2)^k (k!)^{-1} d^k F(z)/dz^k. \tag{4.13}$$

This solution expresses $r^{-k-\frac{1}{2}}q$ in terms of its axial values and holds in the domain of dependence $|z| < 1 - \alpha r$ of the axis. For the cylinder and the sphere, the function $F(z)$ can be approximated in the form

$$F(z) = \mathrm{Re}[A^*(z, n) e^{inf(z)}], \tag{4.14}$$

where $f(z)$ is real, A^* has an asymptotic expansion

$$A^* = \sum_{m=0}^{\infty} A_m^*(z) n^{-m} \tag{4.15}$$

and both the phase and the amplitude coefficients $A_m^*(z)$ are independent of n. We now suppose that $F(z)$ can be approximated in precisely the same way for the containers under consideration. If the functions $f(z)$ and $A_m^*(z)$ are sufficiently smooth, the integral solution (4.12) then yields approximations appropriate respectively to the oscillatory zone, to the quiescent axial zone and to a thin intermediate zone. The axial data function $F(z)$ can be chosen so that the approximation to the integral for the oscillatory zone matches the original modulated waves (4.1); and the remaining approximations to the integral then supply the required approximations for the quiescent zone and the transition zone that match the original modulated waves. The relevant detail is as follows.

From (4.12) and (4.14), we have that

$$q = \tfrac{1}{2} r^{\frac{1}{2}} \mathrm{Re}\left\{\int_{-\pi}^{\pi} A^*(\xi) \exp\{in[f(\xi) - \lambda x]\} \, dx\right\}, \tag{4.16}$$

where

$$\xi = z - \alpha r \cos x. \tag{4.17}$$

The phase $n(f(\xi) - \lambda x)$ in (4.16) is stationary if

$$\alpha r f'(\xi) \sin x = \lambda. \tag{4.18}$$

If the roots $x = x_j$ of equations (4.17) and (4.18) are simple, and $x_i - x_j \ll 1$ for $i \neq j$, the stationary phase approximation yields

$$q \sim (\pi r/2n)^{\frac{1}{2}} \sum_j \mathrm{Re}\{A^*(\xi_j) |\phi_j|^{-\frac{1}{2}} \exp[inf(\xi_j) - ikx_j + \tfrac{1}{4} i\pi s_j]\} + O(n^{-\frac{3}{2}}), \tag{4.19}$$

where

$$\phi_j = (z - \xi_j) f'(\xi_j) + \lambda^2 f''(\xi_j) [f'(\xi_j)]^{-2}, \quad \xi_j = \xi(x_j) \tag{4.20}$$

and $s_j = \mathrm{sgn}\, \phi_j$. Thus the pressure q is represented as a sum of modulated waves. The original assumption that q comprises just two modulated waves in the oscillatory region means that the sum in (4.19) has just two terms (i.e. $j = 1, 2$). (The stationary phase equations (4.17) and (4.18) are equivalent to the former equations (4.8) for the trajectories ξ = const., and the assumption just mentioned implies that where these equations have simple roots $\xi = \xi_j$ they have just two roots.)

By comparing the integral approximation (4.19) for the pressure q to the modulated wave equations (4.1), (4.9) and (4.10), we see that it matches the original representation of q in terms of the modulated-wave approximation to leading order provided that

$$f(\xi_j) = \eta(\xi_j), \quad A^*(\xi_j) = A^0(\xi_j)\left|\frac{2}{\pi}\frac{d\zeta}{d\xi}(\xi_j)\right|^{\frac{1}{2}} e^{\frac{1}{4}\pi i s_j}. \tag{4.21}$$

The amplitude A^* and phase f of the axial data function in the integral are thereby related to the parameters of the original modulated waves (which are presently regarded as known). The equations (4.21) also identify discontinuities in the wave parameters on a trajectory ξ_j = const. at its point of contact $r = r_T(\xi_j), z = z_T(\xi_j)$ with the envelope. On the envelope,

$$z_T = \xi_j - \lambda^2\zeta'(\xi_j)\,\zeta^{-3}(\xi_j), \tag{4.22}$$

so that the ϕ_j in equation (4.19) vanishes and the s_j in equation (4.21) changes sign. Hence, the equations (4.21) imply that

$$\eta_2(\xi) = \eta_1(\xi), \quad A_2^0(\xi) = iA_1^0(\xi), \tag{4.23}$$

where the suffixes 1 and 2 denote the values on a trajectory ξ = const. for points $z > z_T(\xi)$ and $z < z_T(\xi)$ respectively. Thus the phase parameter $\eta(\xi)$ at a point z on the trajectory is unchanged after the point has passed $z = z_T$, whilst the corresponding (complex) amplitude changes by a factor i.

The approximation (4.19), and its consequences, do not apply near the envelope, where the roots $\xi = \xi_j$ of the stationary phase equations nearly coincide. After a short calculation, the appropriate approximation to the integral (4.16) for points near the envelope is found to be

$$q(r, z_T) \sim \pi r^{\frac{1}{2}}\,\mathrm{Re}\{A^*(\xi_T)\,e^{ind}\}\,Ai(c)\,(4nK)^{-\frac{1}{3}}, \tag{4.24}$$

where

$$\xi_T = \xi(r_T(z_T), z_T), \quad r - r_T(z_T) = O(n^{-\frac{2}{3}}) \tag{4.25}$$

and

$$c = K^{-\frac{1}{3}}\,\lambda n^{\frac{2}{3}}(r - r_T(z_T))\,r^{-1}\{1 + (\xi_T - z_T)f''(\xi_T)\,[f'(\xi_T)]^{-1}\}, \tag{4.26a}$$

$$d = f(\xi_T) - \lambda x_T - (r - r_T(z_T))r^{-1}\{[f'(\xi_T)]^2 - \lambda^2\}^{\frac{1}{2}}, \tag{4.26b}$$

$$\lambda^{-1}K = \lambda^2 f'''(\xi_T)\,[f'(\xi_T)]^{-3} + 3(z_T - \xi_T)f''(\xi_T)\,[f'(\xi_T)]^{-1} - 1. \tag{4.26c}$$

After using equations (4.21) to express the f and A^* in equations (4.24) to (4.26) in terms of the parameters η and A^0 of the modulated waves, we get the required approximation for the pressure q in the transition layer that matches the modulated waves. Clearly, the velocity in the transition layer is $O(n^{-\frac{1}{3}})$ and its width is $O(n^{-\frac{2}{3}})$.

For points in the quiescent zone between the transition layer and the z axis, the equations for stationary phase have no roots for ξ. The integral (4.16) can then be estimated simply by integrating by parts. In this way, we find that the pressure

$$q = O(n^{-m}) \tag{4.27}$$

and the velocity is $O(n^{-m+1})$, where m is an integer, provided that the phase $f(\xi)$ of the axial data function and the coefficients $A_s^*(\xi)$, $0 \leqslant s \leqslant m-1$, of the asymptotic series for the associated amplitude A^* have continuous derivatives of orders $m+1$ and

$m-s$ respectively and the $A_m(\xi)$ are bounded. [To express this condition in terms of the modulated waves requires a higher order of matching to the axial data function than has been presented. But it can be anticipated that the condition would hold if the phases $\sigma_j(r,z)$ of the modulated waves and the coefficients of the Poincaré series in n for (a) the radius $r_T(z)$ of the envelope and (b) $(r-r_T(z))^{\frac{1}{4}}\times$ (the amplitudes $A_j(r,z)$ of the modulated waves) are adequately differentiable.] For the sphere and the cylinder, m can be taken to be arbitrarily large.

The integral solution (4.16) applies only in the domain of dependence $|z| < 1-\alpha r$. However, the approximation (4.24) for the transition zone, with $f(\xi)$ and $A^*(\xi)$ expressed in terms of $\eta(\xi)$ and $A(\xi)$, and the connection formulae (4.23) could be derived by the boundary-layer method developed (Buchal & Keller 1960) for the transition layer near a caustic in diffraction theory. So derived, these results are seen to stem from a local approximation which is governed near any point $r = r_T$, $z = z_T$ on the envelope by the modulated wave near that point. Consequently, the results must hold independently of whether the point r_T, z_T is in the domain $|z| < 1-\alpha r$. In the boundary-layer method the pressure for $n \gg 1$ is assumed not to be exponentially large in the axial zone at distances $O(1)$ from the envelope, but this assumption (which serves in place of the condition that the velocity at $r = 0$ is finite) is plausible. From here on, then, we take the transition approximation (4.24) and the connection formulae (4.23) to apply along the entire envelope, irrespective of r_T, z_T being in the domain of dependence of the axis.

For the same reason, the degree of smallness of the pressure q just after crossing the transition zone near the point r_T, z_T on the envelope should not depend on r_T, z_T being in the domain of dependence of the axis. So, we can expect the estimate (4.27) to apply to the whole quiescent zone, providing that the associated conditions on the modulated wave given in parenthesis below (4.27) hold near the entire envelope.

In the sequel some support for the various assumptions made will be got by calculating the modulated waves for axisymmetric containers with certain reflection symmetries and for $\omega \simeq \sqrt{2}\,\Omega$ and $k \ll n$. First, however, we consider the reflection conditions at the boundary and a consequent magnification of the velocity near the critical circles where the characteristic cones of Poincaré's equation (2.4) touch the boundary.

(c) *Reflection conditions*

At the solid boundary, $r = S(z)$ of the oscillatory zone, the condition (2.9) implies that

$$\mathrm{Re}\{A_1 P_1 e^{in\sigma_1}\} = -\mathrm{Re}\{A_2 P_2 e^{in\sigma_2}\}(1+O(n^{-1})), \tag{4.28}$$

where

$$P_j = \rho_j + \alpha^2\zeta_j S'(\xi_j) - \gamma i (nr)^{-1}. \tag{4.29}$$

Hence, the reflection relations for the amplitude and phase are given by

$$|A_1 P_1| = |A_2 P_2|(1+O(n^{-1})), \tag{4.30}$$

$$n\nu_1^* = \pm n\nu_2^* + (\text{an odd integer})\,\pi + O(n^{-1}), \tag{4.31}$$

where $n\nu_j^* = n\sigma_j + \arg A_j + \arg P_j + \frac{1}{2}\pi$. To allocate the appropriate sign to $n\nu_2^*$ in (4.31) we note that

$$\zeta_j = (d\sigma_{bj}/dz)/[1+(\xi_j - z)\,r^{-1}S'(z)], \tag{4.32}$$

where $\sigma_{bj} = \sigma_j(S(z), z)$. Hence, since $\zeta \geqslant 0$ by definition and $\nu_j = \sigma_j$ to leading order, the ambiguous sign in (4.31) is + or − according as the signs of $1+(\xi_j-z)\,r^{-1}S'(z)$ for $j = 1, 2$ are the same or different.

The reflection geometry near a circle C where a characteristic hyperboloid is tangential to the boundary is special and causes an increase in the order of magnitude of the velocity. If a nearly tangential trajectory passes to within a distance $O(\Delta s)^2$ from C it meets the boundary a distance $O(\Delta s)$ from C. Thus the nearly tangential trajectories ($\xi = \xi_1$ say) are more closely spaced than the trajectories ($\xi = \xi_2$ say) that they engender after reflection. This relative proximity causes the gradient of the phase σ_1 normal to the boundary to tend to infinity as $(\Delta s)^{-1}$, as may be seen by using (4.9) and (4.31). In particular, $\zeta_1 = \infty$ on the grazing trajectory ζ_1 = constant. So the tangential hyperboloid degenerates to a characteristic cone of Poincaré's equation.

At first sight, the singularity in $\zeta_1(=\sigma_{1z})$ at the critical circle C seems to make the velocity infinite there. However, this infinity is not realized because of a restriction on the value of $n\sigma+\arg A$ at the critical circle which introduces a compensating zero. Suppose a nearly tangential trajectory $\xi = \xi_1$ near the critical circle $z = z_C$ meets the boundary at the circles A ($z = z_A > z_C$) and B ($z = z_B < z_C$). The denominator in (4.32) can be shown to vary as $z-z_C$ for $|z-z_C| \ll 1$ and $j = 1$ because ζ_1 varies as $|z-z_C|$ and $|S'(z_C)| = 1/\alpha$. So $d\sigma_{1b}/dz$ is $O(1)$ and non-zero at C but changes sign (discontinuously) there. Consequently, opposite determinations of the ambiguous sign in (4.31) apply at A and B. Hence, on combining the reflection relations (4.31) at A and B and letting $A \to B$ (and assuming that the wave parameters A^0, η and ζ vary continuously with ξ_2), we find that

$$\lim_{z_A\to z_C}(n\sigma_1+\arg A_1+\arg P_1)_A+\lim_{z_B\to z_C}(n\sigma_1+\arg A_1+\arg P_1)_B = (\text{odd integer})\,\pi+O(n^{-1}). \quad (4.33)$$

From (4.9), we see that $\nu_{1b}(z_A) \to \nu_{1b}(z_B)$ as $A \to B$, where

$$n\nu_{jb}(z) = n\sigma_{jb}(z)+\arg A_j(S(z),z).$$

Whilst, from (4.29), we find that

$$\lim_{z_A\to z_C} P_1(z_A) = \alpha^2S''(z_C)\,Q-\gamma i(nr)^{-1}, \quad (4.34)$$

$$\lim_{z_B\to z_C} P_1(z_B) = -\alpha^2S''(z_C)\,Q-\gamma i(nr)^{-1}, \quad (4.35)$$

where Q denotes the (finite) value of $\lim_{z_A\to z_C}\zeta_1(\xi_{1A})|z_A-z_C|$. Hence, equation (4.33) implies that

$$n\nu_{1b}(z_C) = N_C\pi+O(n^{-1}), \quad (4.36)$$

where N_C is a large integer of $O(n)$.

The velocity near the critical circle can now be assessed as follows. For points on the trajectory AB near the critical circle, we have from (4.9) and (4.32) that

$$\partial\sigma_1/\partial r = \alpha\zeta_1(1+o(1)), \quad \zeta_1 = [d\sigma_{1b}(z_B)/dz](1+o(1))\,(1+\alpha S'(z_B))^{-1}. \quad (4.37)$$

Thence, to a first approximation, the velocity amplitude on AB becomes

$$(U, V, W) = \left(\frac{n}{r}\right)^{\frac{1}{2}}\left|A_1\frac{d\sigma_{1b}}{dz}\right|_C\frac{\sin n^*Z}{\omega\alpha^2ZS''(z_C)}\left(-i, \frac{2\Omega}{\omega}, \alpha i\right), \quad (4.38)$$

where $Z = z_C-z_B$ and $n^* = nd\nu_{1b}(z)/dz|_{z\uparrow z_C}$. The factor $Z^{-1}\sin n^*Z$ is $O(n)$ when $Z = O(n^{-1})$. Hence, as was foreshadowed above, the velocity is amplified by a factor of $O(n)$ within distances of $O(n^{-2})$ from the critical circle in a direction normal to the boundary.

5. Determination of modes for $k \ll n$ and $\omega \simeq \sqrt{2}\,\Omega$

The determination of the modulated waves (4.1) in the oscillatory zone hinges on the pattern of the ray paths in the axial plane after repeated reflection at the boundary. The relevance of such a pattern is more obvious when the modes are axisymmetric ($k = 0$) and the rays coincide with the characteristic lines $z-\xi = \pm\alpha r$. The quiescent zone is then replaced by a thin axial zone of width $O(n^{-1})$ and the modulated waves can be regarded as being reflected at the axis (Wood 1977*b*). So the pattern in question is the one formed after repeated reflection at the boundaries and the axis. To leading order, the phase is constant along each ray, undergoes a change of $\frac{1}{2}\pi$ on reflection at the axis and is unaltered on reflection at the boundary, save for a change in sign at reflections between either pole ($z = \pm 1$) and its adjacent critical circle. In this case two topologies can be distinguished.

In one of these topologies, the ray pattern contains one or more closed paths – closure being represented by an equation

$$z_i = z_f(z_i, \omega), \tag{5.1}$$

where $z = z_i$ and $z = z_f$ represent the initial and final boundary points of a ray path after a certain number of reflections. The frequencies ω_1 for which this topology occurs generally form continuous bands. In the alternative topology, appropriate to the frequencies ω_2 ($|\omega_2| < 2\Omega$) outside these bands, a continuous function $\Gamma(\omega)$ can be defined and either all or none of the ray paths close depending on whether Γ is or is not a rational number. For these frequencies ω_2, it is conjectured that $\Gamma(\omega_2)$ is not normally constant, so that every eigenfrequency is close to a frequency ω_2 for which all the ray paths close, and that for $k/n \ll 1$, this near closure generally allows a continuous mode to be constructed. The basis for the two topologies is explained in § 6; the basis for the construction is as follows.

For exact closure, with $\omega = \omega^*$ and $k = 0$, the change in phase of one of the characteristics ($j = 1$, say) at the boundary point $z = z_i$ after a cycle of reflections at the boundary and the z axis is of the form

$$n\sigma_{1b}(z_f) - n\sigma_{1b}(z_i) = \tfrac{1}{2}N\pi, \tag{5.2}$$

where N is an integer. (The possibility of a plus sign in place of the minus sign in (5.2) can be discounted on geometrical grounds.) The change in phase after traversing a closed contour must be a multiple of 2π; and this can be achieved by a small change in the frequency ω which slightly displaces the end-point $z = z_f$ of the trajectories formed by repeated reflections, starting with the $j = 1$ trajectory at $z = zi$. The phase change (5.2) after the repeated reflections is unaltered to leading order by a slight change in ω and k/n save possibly for trajectory paths that pass close to the poles ($r = 0, z = \pm 1$) or the critical circles. So, the periodicity condition implies that, to leading order, the change in the phase associated with the slight displacement of z_f is

$$n\sigma_{1b}(z_f(z_i, \omega, \lambda)) - n\sigma_{1b}(z_f(z_i, \omega^*, 0)) = 2\pi \times \text{(integer)}. \tag{5.3}$$

Whence, to a first approximation,

$$\frac{d_1\sigma_b(z_i)}{dz} = 2\pi\times(\text{integer})\left[n(\omega-\omega^*)\frac{\partial z_f(z_i,\omega^*,0)}{\partial\omega}+k\frac{\partial}{\partial\lambda}z_f(z_i,\omega^*,0)\right]^{-1}. \quad (5.4)$$

This equation determines a first approximation to the phase σ_{1_b} at the boundary, and hence to the phases of the modulated waves in the interior for $k/n \ll 1$. The corresponding amplitudes $|A|$ of the modulated waves can be taken to be uniform to a first approximation.

This method is applied below to a class of containers for which exact closure occurs at a particular frequency (when $k = 0$) because of suitable symmetries. The containers in question have meridional sections with mirror symmetry about the lines $z = 0$ and $z = \pm r$. All the ray paths then close when $w = \sqrt{2}\,\Omega$ ($\alpha = 1$) and $k = 0$ after at most three reflections at the boundary, and for $k/n \ll 1$ the modes and the eigenfrequencies near $w = \sqrt{2}\,\Omega$ can be calculated using (5.4) as a first approximation. The restriction (4.36) on the phase at the critical circles and a symmetry condition on the phase at the equator provide the two side conditions needed to specify the eigenfrequencies and fix the arbitrary constant after integrating (5.4).

For azimuthal wavenumbers such that $n \gg k \gg 1$, the trajectories turn rapidly, with a radius of curvature $O(k/n)\,(= O(\lambda))$, at a distance $O(\lambda)$ from the axis. The envelope is then a distance $O(\lambda)$ from the axis, and is approximately the locus

$$z_T = \xi, \quad r_T = \lambda/\alpha\zeta(\xi), \quad (5.5)$$

of the apices of the trajectories ξ = constant.

The branches $z > \xi$ and $z < \xi$ of each trajectory ξ = const. will be called the + and − trajectories respectively and variables associated with either will be distinguished where necessary by a + and − suffix. Away from the axis, where $r^{-1} = O(1)$, the trajectories are given approximately by

$$z-\xi_\pm = \pm\tfrac{1}{2}\alpha r[2-\lambda^2(\alpha r\zeta_\pm)^{-2}]+O(\lambda^4), \quad (5.6)$$

and the phase, amplitude and radial wavenumber $n\rho$ are given by

$$\sigma_\pm = \eta_\pm \mp \lambda^2(\alpha\zeta_\pm r)^{-1}+O(\lambda^4), \quad A_\pm = A^{**}e^{\pm i\pi/4}+O(\lambda^2)+O(n^{-1}), \quad (5.7)$$

$$\rho_\pm = \mp\alpha\zeta_\pm[1-\tfrac{1}{2}\lambda^2(\alpha\zeta_\pm r)^{-2}]+O(\lambda^4), \quad (5.8)$$

where A^{**} is a constant along each trajectory $\xi_+ = \xi_-$. To leading order, the phase σ and the complex amplitude A are constant along each + or − trajectory, save where $r = O(\lambda)$, but the amplitude changes by a factor i on passage round the apex of any trajectory from points where $r \gg \lambda$ on the negative branch to points where $r \gg \lambda$ on the positive branch.

We consider next the reflection relations (4.28) to (4.31) for $\lambda \ll 1$ and $r^{-1} = O(1)$. From (4.29), (4.32) and (5.8), we get

$$P_\pm = \mp\alpha\, d\sigma_{b\pm}/dz-\gamma i(nr)^{-1}+O(\lambda^2). \quad (5.9)$$

The phase relation (4.31) implies that $d\sigma_{b+}/dz = d\sigma_{b-}/dz+O(n^{-1})$, where the suffix b again denotes the value at the boundary $r = S(z)$. Hence the reflection relation for the magnitude of the amplitude reduces to

$$|A_+| = |A_-|(1+O(n^{-1})+O(\lambda^2)). \quad (5.10)$$

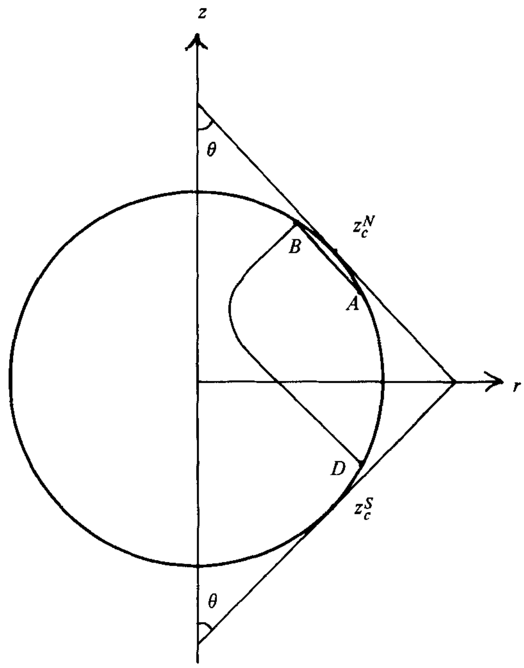

FIGURE 2. Characteristics (AB and BD), of the phase equation (4.6), used in determining the eigenfrequencies.

The quantities $\arg P_\pm$ that occur in the reflection relations (4.31) are given by

$$\arg P_+ = \begin{cases} O(\lambda) & \text{for} \quad z < z_c^S, \\ \pi + O(\lambda) & \text{for} \quad z > z_c^S, \end{cases} \tag{5.11}$$

$$\arg P_- = \begin{cases} O(\lambda) & \text{for} \quad z < z_c^N, \\ \pi + O(\lambda) & \text{for} \quad z > z_c^N, \end{cases} \tag{5.12}$$

where $z = z_c^S, z_c^N$, with $z_c^S < z_c^N$, are the values of z at the two critical circles.

To determine the phase we consider the path, after one reflection, of a negative trajectory which meets the boundary at a circle A with co-ordinates r_A, z_A, where $0 < z_A < z_c^N$. This negative trajectory meets the boundary at a circle r_B, z_B, where $z_B > z_c^N$, and in general reflects as a positive trajectory which meets the boundary at a circle D with co-ordinates r_D, z_D, where $0 > z_D > z_c^S$ (figure 2). If $\lambda = 0$ and $\theta = \frac{1}{4}\pi$ ($\alpha = 1$), the trajectories reduce to the cones $z = \pm r + \text{constant}$, and because of the container's symmetries the circles A and D are mirror images in the plane $z = 0$. For small λ and $\theta \simeq \pi/4$, a straightforward perturbation calculation shows that

$$-z_D - z_A = \Delta_0\,\delta + \Delta_1\,\delta^2 + O(\delta^3), \tag{5.13}$$

where $\delta = \alpha - 1 \ll 1$,

$$\Delta_0 = 2(r_A + z_A S')/(1 - S'^2), \tag{5.14a}$$

$$\Delta_1 = \frac{S'}{(1-S'^2)^2}\{(r_A - z_A)(1-S')(3+S') + (r_A + z_A)(1+S')^2\}$$
$$+ \frac{S''}{2(1-S')}\left\{\Delta_0^2 - \frac{2(r_A - z_A)^2}{(1+S')^3}\right\} + \frac{\lambda^2}{\delta^2[d\sigma_{b-}(z_A)/dz_A)]^2}\{z_A^{-1}S' - r_A^{-1}\} \tag{5.14b}$$

and the S' and S'' here are evaluated at $z = z_A$. The expansion in δ appears to be regular for fixed z_A, $0 < z_A < z_c^N$, providing the container is suitably smooth.

The container's symmetry about $z = 0$ obviates the need to follow the reflections of the characteristics generated by AB till one of the characteristics returns to the boundary near A. Because of this symmetry, q is either an odd function or an even function of z, so that either $q = 0$ or $q_z = 0$ on $z = 0$. Consequently, on $z = 0$,

$$|A_+| = |A_-|, \quad n\nu_{b+} = -n\nu_{b-} + N^*\pi, \tag{5.15}$$

where N^* is an odd or even integer according as q is an odd or even function of z. The negative sign is appropriate in (5.15) because $(\sigma_{r+})(\sigma_{r-}) \leqslant 0$ on $z = 0$ (cf. (4.9)). It follows from the characteristic relations (4.9) that

$$|A_+(r,z)| = |A_-(r,-z)|, \quad n\nu_+(r,z) = -n\nu_-(r,-z) + N^*\pi, \tag{5.16}$$

where $n\nu_\pm = n\sigma_\pm + \arg P_\pm$. In particular

$$n\nu_{b+}(z_D) = -n\nu_{b-}(-z_D) + N^*\pi, \tag{5.17}$$

where $n\nu_{b\pm}$ represents the value of the phase $n\nu_\pm$ at the boundary. Since $-z_D - z_A$ is small, the phase difference $-\nu_{b+}(z_D) - \nu_{b-}(z_A)$ can be approximated by

$$-\nu_{b+}(z_D) - \nu_{b-}(z_A) = \frac{d\nu_{b-}(z_A)}{dz_A}(-z_D - z_A) + \frac{1}{2}\frac{d^2\nu_{b-}(z_A)}{dz_A^2}(z_D + z_A)^2 - N^*\pi n^{-1} + O(\delta^3). \tag{5.18}$$

After combining the changes in phase that occur along the characteristics AB and BD and on reflection at B and D, we find that

$$-n\nu_{b+}(z_D) - n\nu_{b-}(z_A) = M\pi + L + O(\lambda) + O(n^{-1}), \tag{5.19}$$

where

$$M = k + \tfrac{1}{2} + 4N_1 + 2N_c - N^*, \tag{5.20}$$

$$L = 2\lambda k[r_A^{-1} - z_A^{-1}S'(z_A)]/d\sigma_{b-}(z_A)/dz_A. \tag{5.21}$$

The N_1 in (5.20) is such that $2N_1 - 1$ is the arbitrary odd integer in the phase relation (4.31). The leading terms of (5.13), (5.18) and (5.19) yield the approximation

$$d\nu_{b-}(z)/dz = M\pi/n\delta\Delta_0(z), \tag{5.22}$$

which, together with (5.14), gives

$$n\sigma_{b-}(z_A) = \frac{M\pi}{2\delta}\int_0^{z_A} \frac{1 - [S'(z)]^2}{zS'(z) + S(z)}\,dz + (N_1 + \tfrac{1}{2}N^*)\,\pi. \tag{5.23}$$

This equation determines the phase σ at the boundary to a first approximation apart from certain constants. To encompass significant proportional changes in k, the number M defined by (5.20) must be at least as large as $O(k)$. Accordingly, we allow δ to be as large as $O(\lambda)$. The phase σ at interior points can be calculated from the relations (4.9) and its boundary values (5.23).

The boundary of the quiescent zone is now determined to a leading order by

$$z_T = z_A - S(z_A), \quad r_T = 2\delta k\{z_A S'(z_A) + S(z_A)\}/\{M\pi[1 + S'(z_A)]\}, \tag{5.24}$$

as may be inferred from (4.37), (5.5) and (5.23). For example, the boundary of the quiescent zone for the sphere ($S(z) = (1-z^2)^{\frac{1}{2}}$) becomes

$$z_T^2 + (M\pi r_T/2\delta k)^2 = 2, \tag{5.25}$$

which is consistent with the result (3.20) got from the exact solution.

The procedure by which the approximation (5.23) to the phase was obtained does not apply to points on the boundary near $z = 0$. The negative trajectories that meet the boundary near $z = 0$ also meet the boundary near the z axis and here they curve rapidly and, if near enough to the caustic, they reflect as negative trajectories rather than as positive trajectories. The rapid curvature invalidates the geometrical approximation for the deviation (5.13) of $-z_D$ from z_A and back reflection as a negative trajectory invalidates the phase reflection relations used. Despite these defects in the derivation, however, the result (5.23) appears to be correct for points z_A near $z = 0$, provided that the container has a high enough order of contact with the sphere $r^2+z^2 = 1$ near the z axis. For the sphere, the phase ν_- at the boundary is given by

$$n\nu_{b-} = \text{constant} \pm n_0 \cos^{-1}[n_0 z(n_0^2-k^2)^{-\frac{1}{2}}] \mp k\cos^{-1}[kz(n_0^2-k^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}], \tag{5.26}$$

where $n_0 = n+\frac{1}{2}$. The expansion of the variable part of this expression in powers of n^{-1} and of $\lambda = k/n$ for large n and small λ converges uniformly for $1 < z < 1$. Pursuing further the asymptotic expansions of the exact solution (3.8) for the sphere, we find that $d\nu_{b-}/dz$ has an expansion in n^{-1} and in λ which is uniform for $-1 < z < 1$, and δ has an expansion in powers of n^{-1} and λ. For the class of containers being considered in this section the ray procedure can be expected to generate expansions for $d\nu_{b-}/dz$ in n, λ and δ which are uniform in z for $0 < z < z_c^N$. Moreover $\nu_{b-}(z_A)$ is thereby determined apart from constants by the geometry of the container near z_A, z_B and z_D *independently of the geometry elsewhere.* A reconstructed procedure for finding $\nu_{b-}(z_A)$ when z_A is small, with suitably amended formulae to replace the original estimate of $-z_D-z_A$ and the original reflection relations, would be expected to retain this property. So, if the sphere has contact of a sufficiently high order near $r = 0$ and $z = 0$ with the container under consideration the (truncated) asymptotic expansion of $\nu'_{b-}(z_A)$ in n, λ and δ for the container should be the same near $z_A = 0$ as for the sphere. The case of the sphere shows this expansion to be uniform with respect to z near $z = 0$ after δ had been replaced by its expansion in powers of n^{-1} and λ. Consequently, unless the relation of δ to λ for the sphere happens to remove non-uniformities in the expansion, the expansion must also be uniform near $z = 0$ for the sphere. We shall assume that this removal of non-uniformities does not in fact occur and that the ray procedure for $0 < z < z_c^N$ does in fact generate the correct expansion in n^{-1}, λ and δ for $0 \leqslant z < z_c^N$. The question of how close the contact of sphere and container near $z = 0$ needs to be to achieve a given accuracy in the expansion is left unsettled. (The symmetry of the container ($r = S(\pm z)$, $z = \pm S(r)$) ensures that the contacts near $r = 0$ and $z = 0$ are equally precise.) The ray procedure also needs scrutiny when z_A is near the critical circle $z = z_c^N$, but, so far as it has been taken, the procedure appears to be self-consistent in this neighbourhood.

Thus far the parameter δ which defines the eigenfrequency has remained unspecified. To determine δ, we impose the condition that

$$n\nu_{b-}(z_c^N) - n\nu_{b-}(0) = \tfrac{1}{2}M_0\pi + o(1), \tag{5.27}$$

where $M_0 = 2N_c - N^* - N_1$. The value of ν_{b-} at the critical circle was given by (4.36).

Its value at $z = 0$ follows from (4.31), (5.12) and (5.17), the odd integer in equation (4.31) being designated as $2N_1 - 1$ for the range $z_c^S < z < z_c^N$ between the critical circles. To apply the condition (5.27) we need to estimate $n\nu_{b-}(z)$ with an error $o(1)$. On returning to the equations (5.13), (5.18) and (5.19) and substituting where necessary for σ'_{b-} and σ''_{b-} from the first approximation (5.22), we get

$$n\nu_{b-}(z_A) = \delta^{-1}\int^{z_A}(M\pi + L)\,\Delta_0^{-1}\,dz + M\pi\left\{\tfrac{1}{2}\log\Delta_0(z_A) - \int^{z_A}\Delta_1\Delta_0^{-2}\,dz\right\} + o(1). \quad (5.28)$$

Thence,

$$\delta = 2M\left[\int_0^{z_c^N}\Delta_0^{-1}\,dz\right]\Big/\{M_0 + M\log(2/\Delta_0(z_c^N))\}$$
$$+2\int_0^{z_c^N}[M\Delta_1\Delta_0^{-2} - 2L(\pi\delta)^{-1}]\,\Delta_0^{-1}\,dz. \quad (5.29)$$

The eigenfrequency ω is related to δ by

$$\omega = \surd 2\,\Omega[1 - \tfrac{1}{2}\delta + \tfrac{1}{8}\delta^2 + O(\delta^3)], \quad (5.30)$$

and this relation together with (5.14) and (5.29) determines the eigenfrequencies near $\surd 2\,\Omega$ in terms of the shape $r = S(z)$ of the boundary. The leading approximation

$$\omega = \surd 2\,\Omega\left[1 - MM_0^{-1}\int_0^{z_c^N}\Delta_0^{-1}\,dz\right], \quad (5.31)$$

in which the arbitrary integers M_0 and M are $O(n)$ and $O(k)$, respectively, shows that the distribution of the values of ω near $\surd 2\,\Omega$ is similar to that of the rational numbers.

To a first approximation the amplitude parameter $|A_0|$ is constant along the trajectories and remains constant on reflection at the boundary. Hence $|A_0|$ is approximately constant throughout the oscillatory region, save near the caustic where the amplitude equation (4.7) fails. It follows from (4.9) that the amplitude $|A|$ is approximately constant in the oscillatory region save where $r = O(\lambda)$. The breakdown in the ray procedure for negative trajectories meeting the boundary near the axis $z = 0$ is not expected to impair this conclusion, provided the container and sphere have a close enough contact near $z = 0$, for the same reasons as were advanced in relation to the corresponding breakdown for the phase.

It might be added that the above results for the eigenfrequencies, the phase σ at the boundary and the amplitude A agree with their counterparts got from the exact solution for the sphere.

6. Dichotomy of modes in two dimensions and for $k \ll n$

The distinction between modes referred to above is fundamental and is worth elaborating.

We begin by returning to two-dimensional modes in smooth, convex rigid containers, mentioned in the introduction, for which a similar but sharper distinction can be drawn. The distinction follows from a consideration of the successive reflection points at the boundary of a characteristic $x - \alpha z =$ constant, say, that leaves the boundary at a point P_0 (and reflects as a characteristic $x + \alpha z =$ constant, and so on). Let $P_1, P_2, \ldots, P_n$ denote every second reflection point and let $\theta_0, \theta_1, \ldots, \theta_n$ denote the corresponding polar angles defined cyclically (with the co-ordinate origin O in the fluid) so that $\theta_n < \theta_{n+1} < \theta_n + 2\pi$. Then the relation $\theta_1 = \theta_1(\theta_0, \alpha)$ for varying θ_0 defines a smooth, one–one sense-preserving map of the boundary on to itself. For a sufficiently smooth boundary several results, listed below, ensue. The first two of these are merely preliminary. The next two underlie more directly the distinction in question.

(i) There is a number

$$\Gamma(\alpha) = \lim_{n\to\infty} \theta_n/2\pi n$$

whose value is independent of the initial point P_0 (Coddington & Levinson 1955). The number Γ depends here on the ray gradient $\alpha(\omega)$, and hence on the frequency ω, and the shape of the container.

(ii) The number $\Gamma(\alpha)$ varies continuously with α. (This follows from a result stated by Poincaré, which is proved by Arnol'd (1965).)

When $\Gamma(\alpha)$ is a rational number (m/n, say, where m and n are co-prime), then either

(1) there is a finite number n of discrete characteristic paths that close after $2n-1$ reflections (so that $P_n = P_0$),

or

(2) every characteristic path closes after $2n-1$ reflections regardless of its initial point P_0 (John 1941). If $\Gamma(\alpha)$ is irrational no characteristic paths close.

Closure is represented by the equation

$$\theta_0 = \theta_n(\theta_0, \omega). \tag{6.1}$$

Since the gradients $\pm\alpha(\omega)$ of the characteristics vary smoothly with ω (for $|\omega| < 2\Omega$) and the boundary is smooth, the end value θ_n varies smoothly with ω and θ_0 over finite ranges of each. So, if (6.1) has discrete roots for θ_0, they generally occur for continuous ranges of ω; which means that the discrete closure topology (1), if it occurs, generally occurs for one or more continuous frequency bands. Within each band n remains fixed. Hence $\Gamma(\omega)$ also remains fixed, since it is necessarily a rational number. We shall presume that $\Gamma(\omega)$ is not normally constant for the frequencies ω_2, $|\omega_2| < \Omega$, outside these bands. Every ray path then closes for a densely distributed subset ω_2^* of the frequencies ω_2.

In the case of discrete closure, the characteristic paths, when continued in either direction, ultimately converge to one or other of the closed paths, and this convergence causes an infinity in the gradient of the stream function normal to the closed path. Moreover a narrow enough pencil of neighbouring paths does not retrace or overlap its earlier parts as it cycles (in either direction), so that the values ψ_0 of the stream function $\psi = \psi_0(x-\alpha z) - \psi_0(x+\alpha z)$ can be assigned arbitrarily to each path of the pencil. Thus, the corresponding modes occur for discrete frequency bands, have infinite internal velocities on the closed paths and are in part indeterminate. Modes in effect of this type were discussed by Stewartson (1971, 1972) for oscillations trapped near the equator of a thin spherical annulus, the oscillations being two-dimensional within the approximation he considered.

In the second topology the values ψ_0 can be assigned arbitrarily to each closed path. So the corresponding modes are indeterminate and can be continuous.

The ray patterns of the axisymmetric modulated waves can be classified in the same way. For $k = 0$, the rays in an axial plane are the lines $z \pm \alpha r =$ constant and they effectively reflect at the axis, as was noted above. This reflection can be represented by using the two halves ($\phi = 0$, $\phi = \pi$, say) of an axial cross-section and continuing each ray straight across the axis. The pattern of rays after successive reflections at the boundary and the axis then becomes identical with that (after boundary reflections only) in a two-dimensional flow with rays $z \pm \alpha x =$ constant and a boundary $x = \pm S(z)$. Hence, a rotation number $\Gamma(\alpha)$ can be defined just as in two dimensions and precisely the same classification of frequencies and closed paths follows.

Small changes in k/n affect the ray geometry only slightly. If discrete closure occurs for $k = 0$ it can be expected to occur at the same frequency for small enough k/n, provided that $\theta_n(\theta_0, \omega, k/n)$ varies sufficiently smoothly with k/n, since the roots θ_0 of the equation (6.1) can then be expected to vary continuously with k/n. The previous reasoning concerning the indeterminacy and singularity of the stream function in the case of discrete closure in two dimensions now applies to the phase σ of the modulated waves: whilst, to a first approximation, their amplitude $|A|$ can be taken to be uniform. Thence, we infer that for $k/n \ll 1$ (and $k \gg 1$), there are in general discrete frequency bands in the range $|\omega| < 2\Omega$ for which a finite number of the ray paths close after a sufficient number of reflections.

Hence for $k/n \ll 1$, as in two dimensions, there are in general continuous bands of frequencies for which a finite number of ray paths close and there are (apparently) possible modes corresponding to each of these frequencies that have infinite velocities on the closed paths and are in part indeterminate.

For small enough values of k/n, every frequency ω_2 ($|\omega_2| < 2\Omega$) outside all these bands must be close to a value for which all the ray paths close when $k = 0$. The method for determining the modes proposed in § 5 then applies and there is a prospect, subject to qualifications noted below, of continuous, well-determined modulated-wave modes for a densely distributed subset of the frequencies ω_2.

7. Conclusions

The explicit solutions for free oscillations in the cylindrical cans and in the sphere show that the modes with a large azimuthal wavenumber k and a large meridional wavenumber n comprise modulated waves (as represented in (4.1)). These waves occupy the whole flow except for a quiescent zone about the axis of width $O(k/n)$. The velocities are in the main of a uniform order of magnitude ($O(1)$ say) in the modulated wave and are smaller by a factor of exponential order ($e^{-|O(n)|}$) in the quiescent zone. (When $k = O(1)$ and $n \gg 1$, the quiescent zone is replaced by a thin zone of width $O(n^{-1})$ in which the velocity is amplified to $O(n^{\frac{1}{2}})$ by radial focusing (Wood 1977*b*).) When $k/n \simeq 1$, the modulated waves in the sphere are confined to the neighbourhood of the equator, the eigenfrequency ω being then $\ll \Omega$.

The solutions for the cylindrical annular containers conform with one's natural expectation that the imposition of an internal boundary into the quiescent zone (or any other alteration of the solid boundary of this zone) would only slightly perturb the associated modulated wave. Consequently, the spherical annulus in particular can be expected to have modulated-wave modes almost identical to those in the sphere, provided that the inner sphere lies wholly in the quiescent zone.

The velocity of the modulated waves in a sphere is magnified by a factor $O(n)$ in the neighbourhood of the critical circles (where the characteristic cones $r \pm \alpha z =$ constant touch the sphere). The magnification is caused by the relative crowding of the ray paths in this neighbourhood after being reflected nearly parallel to the boundary. A similar amplification was found by Stewartson & Rickard (1969) for modes in a thin spherical annulus. In their analysis, the velocity was infinite on the characteristic cones that touch the inner sphere. (Subsequent papers by Walton (1975) and by Stewartson & Walton (1976) allowed for viscosity and stratification.) The velocity at the critical circles C is finite here, because the phases σ at C take values that, in the absence of crowding, would produce a node there; and the zero thus induced cancels the infinity that would otherwise be produced by the crowding of characteristics.

The postulate that modulated-wave modes represented as in (4.1) exist in smooth, convex, axisymmetric containers of arbitrary shape leads to a similar quiescence near the axis and velocity magnification by a factor $O(n)$ near the critical circles. Indeed a quiescent axial zone is a general characteristic of modulated waves varying as $\exp[i(k\phi+n\Sigma(r,z))]$ $(k,n \gg 1)$ and propagating relative to a steady, homogeneous medium that is isotropic in planes z = constant (so that their dispersion relations have the same form $D(\Sigma_z,(\nabla\Sigma)^2-\Sigma_z^2,\omega)=0$ as for inertial waves). In all such cases, the ray paths of the component of the group velocity in the meridional plane are hyperbolic and the width of the quiescent zone is $O(k/n)$, provided that $(\nabla\Sigma)^2-\Sigma_z^2=O(1)$. The velocity amplification near the critical circles C occurs in the same way as for a sphere, though the values of the phases σ at C, which serve to avert an infinity in the velocity, are now derived on the *a priori* assumption that the modulated wave incident obliquely at C is continuous. The approximation (4.24) derived for the transition region between the quiescent zone and the modulated wave shows this region to have a width $O(n^{-\frac{2}{3}})$ and velocities typically $O(n^{-\frac{1}{3}})$ times those in the modulated wave.

The amplitude and the phase at the boundary and the eigenfrequencies ω were determined explicitly for $k/n \ll 1$, $\omega \simeq \sqrt{2}\,\Omega$ and for containers whose meridional cross-sections have mirror symmetry about the lines $z=0$ and $z=\pm r$; and the results obtained check with those derived from the exact solutions for the sphere. The eigenfrequencies are distributed like rational numbers. For $k/n \ll 1$ (with $k \gg 1$), the method used to determine the modulated waves promises to apply, in principle, to smooth, convex axisymmetric containers of arbitrary shape and to the whole of the frequency range for which continuous modes appear admissible. When k/n is not small, the determination of the modes presents unresolved difficulties. Correspondingly, there is less support when k/n is not small, than when $k/n \ll 1$, for the assumption that modulated wave modes represented as in (4.1) exist in non-cylindrical and non-spherical geometries.

When k/n is small enough, the frequency range $|\omega| < 2\Omega$ divides into two parts. One part comprises one or more continuous bands for which a finite number of the ray paths in an axial plane close after sufficiently many reflections at the boundary. For each frequency in these bands there are apparently modes which have infinite velocities on the closed paths and which are in part indeterminate.

For a given small value of k/n every value of the frequency ω_2 ($|\omega_2| < 2\Omega$) outside these frequency bands is generally close to a value (ω_0) for which all the ray paths close when $k=0$. This near closure of all the ray paths allows the rate of change in the phase σ of (one of the two families of) the modulated waves around the boundary to be related (as in 5.4) to the geometry of the ray paths, to the frequency increment $\omega-\omega_0$ and to k/n. A similar relation holds for the amplitude $|A|$ of the (radially scaled pressure q of the) waves, which happens to be uniform to first order for small k/n. The amplitudes and phases so determined are continuous save possibly near the poles and the critical circles. Thus, continuous modulated waves, like those in a sphere, can be constructed provided that singularities near the poles and the circles can be avoided. In the explicit determination, made in §5 and noted above, singularities near the pole were obviated by assuming a sufficiently close contact of the container with the sphere $r^2+z^2 = 1$ near the poles and exploiting the known continuity of the modes in a sphere. The argument used in this connection relied on the behaviour of the waves near the poles being closely related to the local geometry and it can be expected to generalize to other containers. An infinity in velocity at the critical circles was avoided by means of a suitable choice of phase (4.36), which helped fix the eigenfrequencies, and no other singularities at the critical circles were then apparent to the accuracy of the approximation used. Whether the eigenfrequencies can be similarly specified and singularities at the critical circles avoided in configurations with more complex closed ray paths when $k = 0$ requires further study.

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